

Born-Infeld extension of Lovelock brane gravity

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Abstract. We present a Born-Infeld type model to describe the evolution of p -dimensional branes propagating in a flat Minkowski spacetime which, when expanded explicitly, it gives rise to a finite series involving $(p + 1)$ geometrical terms that are related to the Lovelock brane invariants. This model is a second-order volume element that depends on the intrinsic and the extrinsic geometry of the worldvolume swept out by the brane, and it can be regarded as a deformation of the minimal volume element. The field equations are of second-order and we express these in terms of conserved brane tensors. Contrary to the Lovelock theory in gravity, the number of Lovelock brane Lagrangians differs in this case, and it only depends on the dimension of the worldvolume, reflecting the fact that the embedding functions are the field variables instead of the metric. Moreover, we also provide a number of classically equivalent actions for this BI type action and discuss their Weyl invariance in any dimension which naturally requires the introduction of some auxiliary fields.

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1. Introduction

It is argued that the most general effective action governing the evolution of a brane propagating in a fixed background spacetime of dimension N is given by a local action involving a linear combination of higher curvature terms constructed using scalars characterizing the geometry of the worldvolume swept out by the brane [1, 2]. However, when looking for both worldvolume reparametrization and background spacetime diffeomorphism invariances these geometrical scalars are limited. At lowest order we have the ordinary Dirac-Nambu-Goto (DNG) action which is proportional to the volume swept out by the brane. The leading order correction term is quadratic in the extrinsic curvature of the worldvolume and it was originally proposed by Polyakov as a QCD stringy description [3]. Unfortunately, a clumsy fact is that the associated equations of motion are in general fourth-order equations in the field variables. Nonetheless, not all these geometrical scalars fall into this category. For a given dimension p we can pick up a special subset of second-order correction terms that stabilize the brane dynamics having potential physical applications. These are related either to the counterterms

of possible bulk Lovelock invariants [4] or to the Lovelock brane terms, which renders a given system free from many of the pathologies that plague higher-order derivative theories (see also [5] for a review on this topic.). This fact is important because it assures no propagation of extra degrees of freedom. Interest in Lovelock brane theory has attracted quite a lot of attention recently for its rich structure and applications in cosmic acceleration [6, 7, 8].

On the other hand, the usefulness of the Born-Infeld structures have grown by leaps and bounds over the past few years because of their attractive geometric properties, and capacity to implement some finite bounds in several physical theories [9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. These BI structures find a natural place in the brane context since they can encode dynamic symmetries thus providing physical implications. Some related constructions ignoring electromagnetism, but instead including higher-order brane curvature terms, have been proposed in [19, 20]. In this sense, we can conjecture on the possibility that a BI structure may contain intrinsically to the Lovelock brane theory. Thereby, we can ask if there exists a covariant determinantal action which may reproduce to the Lovelock brane invariants under an expansion of such alternative volumen element. This is the issue we will focus on this work.

Our purpose in this paper is twofold. In the first place, we provide a robust Lovelock description for p -dimensional branes, evolving in a flat Minkowski spacetime. We show that the Lovelock brane invariants can all be obtained by using antisymmetric products of the extrinsic curvature and the Gauss-Codazzi integrability condition for surfaces. We obtain the general equations of motion in terms of conserved tensors. In the second place, in the spirit of [12], we propose a second-order BI type action for codimension one branes floating in a flat Minkowski spacetime. We find that the expansion of this alternative volume element, casts out the brane Lovelock invariants which can be all collected in a finite series. In our approach we consider only as necessary the functional dependence on both the first and second fundamental forms g_{ab} and K_{ab} , respectively, inherited by the brane trajectory. Finally, to complement this BI type approach, we provide a number of classically equivalent actions for this model and we discuss their Weyl invariance through the introduction of an auxiliary metric and a scalar field. These actions could be useful in the path integral treatment for branes described for a BI type action as well as the derivation of their critical dimension. One of the novelties in our Lovelock brane description is the implementation of the so-called *conserved stress tensor* [21]. The mechanical content for Lovelock branes will be obtained effortlessly through the use of this conserved stress which supplies the dynamical information of the geometrical degrees of freedom associated to these branes despite the higher-order dependence of the theory. This tool is nothing but the Noether current associated with the translation invariance of the action.

The paper is organized as follows. In Section 2 we develop the Lovelock theory for branes evolving in a flat Minkowski spacetime. We obtain for illustration several Lovelock brane Lagrangians. We obtain the conserved brane tensors useful to express the resulting equations of motion. These tensors are in relation to the associated Noether

charge of the Lovelock terms. In Section 3 we introduce a BI type action containing the induced metric and the extrinsic curvature whose curvature expansion leads to the Lovelock brane Lagrangians. Obviously, one can also consider matter couplings in this description but it is left for future work. In Section 4 we provide some classically equivalent actions for this BI type model and discuss their symmetries. We conclude in Section 5 with some comments and discuss our results. Appendix A and B gather information about standard Lovelock theory notation and some mathematical relations useful for expanding the BI like-structures.

2. Lovelock brane theory

Consider a spacelike brane denoted by Σ of dimension p floating in a $N = (p + 2)$ -dimensional flat Minkowski background spacetime \mathcal{M} with metric $\eta_{\mu\nu}$ ($\mu, \nu = 0, 1, \dots, p+1$). Σ sweeps out an oriented hypersurface manifold of dimension $p+1$, known as worldvolume and denoted by m , described by the embedding functions $x^\mu = X^\mu(\xi^a)$ where x^μ are local coordinates of \mathcal{M} and ξ^a are local coordinates of m , and X^μ are the embedding functions ($a, b = 0, 1, \dots, p$). The only geometrically significant derivatives of X^μ are encoded in the induced metric tensor $g_{ab} = \eta_{\mu\nu} e^\mu_a e^\nu_b := e_a \cdot e_b$ and the extrinsic curvature tensor $K_{ab} = -n \cdot \nabla_a e_b = K_{ba}$ where $e^\mu_a = \partial_a X^\mu$ are the tangent vectors to m , and n^μ denotes the normal vector to the worldvolume and ∇_a is the covariant derivative compatible with the induced metric g_{ab} .

For a $(p+1)$ -dimensional worldvolume the most general action whose field variables are the embedding functions and which maintain the field equations of motion of second order is given by

$$S[X] = \int_m d^{p+1}\xi \sqrt{-g} \sum_{n=0}^{p+1} \alpha_n L_n(g_{ab}, K_{ab}), \quad (1)$$

where

$$L_n(g_{ab}, K_{ab}) = \delta_{b_1 b_2 b_3 \dots b_n}^{a_1 a_2 a_3 \dots a_n} K^{b_1}_{a_1} K^{b_2}_{a_2} K^{b_3}_{a_3} \dots K^{b_n}_{a_n}, \quad (2)$$

and $\delta_{b_1 b_2 b_3 \dots b_n}^{a_1 a_2 a_3 \dots a_n}$ denotes the generalized Kronecker delta (gKd) (see Appendix A for details and conventions), $g = \det(g_{ab})$ and α_n are constants with appropriate dimensions. Here, we set $L_0 = 1$. This action is invariant under reparametrizations of the worldvolume. The Lagrangian (1) is a polynomial of degree $n \leq p + 1$ in the extrinsic curvature and hence the action (1) is a second-order derivative theory. By construction, these Lovelock brane terms vanish for $n > p + 1$ whereas the term with $n = p + 1$ corresponds to a topological invariant not contributing to the field equations. For example, for the relativistic string case with $p = 1$ the Lovelock term $L_2 = \mathcal{R}$ (see below for details) corresponds to a topological invariant because of the Gauss-Bonnet theorem. Note the resemblance with the original Lovelock gravity framework because in such a case we have powers of the spacetime Riemann tensor. Since the independent variables to describe the worldvolume are the embedding functions instead of the metric, we then have one

greater number of Lovelock terms contrary to the gravity case. By expanding out Eq. (2) in terms of minors we have

$$\begin{aligned} L_n &= \left[\delta_{b_1}^{a_1} \delta_{b_2 b_3 b_4 \dots b_n}^{a_2 a_3 a_4 \dots a_n} - \delta_{b_2}^{a_1} \delta_{b_1 b_3 b_4 \dots b_n}^{a_2 a_3 a_4 \dots a_n} + \delta_{b_3}^{a_1} \delta_{b_1 b_2 b_4 \dots b_n}^{a_2 a_3 a_4 \dots a_n} + \dots + (-1)^{n-1} \delta_{b_n}^{a_1} \delta_{b_1 b_2 b_3 \dots b_{n-1}}^{a_2 a_3 a_4 \dots a_n} \right] \times \\ &\quad K^{b_1}_{a_1} K^{b_2}_{a_2} K^{b_3}_{a_3} \dots K^{b_n}_{a_n}, \\ &= K L_{n-1} - (n-1) \delta_{b_2 b_3 b_4 \dots b_n}^{a_2 a_3 a_4 \dots a_n} K^{b_2}_c K^c_{a_2} K^{b_3}_{a_3} \dots K^{b_n}_{a_n}, \end{aligned} \quad (3)$$

where we have relabeled some indexes and used the antisymmetric properties of the gKd. The iterative expansion of the remaining gKd in the latter equation yields

$$\begin{aligned} L_n &= K_{ab}^1 L_{n-1} - (n-1) K_{ab}^2 L_{n-2} + (n-1)(n-2) K_{ab}^3 L_{n-3} + \dots \\ &\quad + (-1)^{n-1} (n-1)(n-2)(n-3) \dots 2 \cdot 1 \cdot K_{ab}^n L_0, \end{aligned}$$

where we have introduced the useful short hand notation: $K_{ab}^1 := K^a_a, K_{ab}^2 := K^a_b K^b_a, K_{ab}^3 := K^a_b K^b_c K^c_a$ and so on. The terms in the previous equation can be all collected in the expression

$$L_n = \sum_{r=1}^n \frac{(-1)^{r-1} (n-1)!}{(n-r)!} K_{ab}^r L_{n-r}, \quad (4)$$

where $n = 1, 2, 3, \dots, p+1$. From the recursion formula (4) we can compute the first Lovelock brane Lagrangians that are explicitly listed below

$$L_0 = 1, \quad (5a)$$

$$L_1 = K, \quad (5b)$$

$$\begin{aligned} L_2 &= K^2 - K_{ab}^2, \\ &= \mathcal{R}, \end{aligned} \quad (5c)$$

$$L_3 = K^3 - 3K K_{ab}^2 + 2K_{ab}^3, \quad (5d)$$

$$\begin{aligned} L_4 &= K^4 - 6K^2 K_{ab}^2 + 8K K_{ab}^3 + 3(K_{ab}^2)^2 - 6K_{ab}^4, \\ &= \mathcal{R}^2 - 4\mathcal{R}_{ab} \mathcal{R}^{ab} + \mathcal{R}_{abcd} \mathcal{R}^{abcd}, \end{aligned} \quad (5e)$$

$$\begin{aligned} L_5 &= K^5 - 10K^3 K_{ab}^2 + 20K^2 K_{ab}^3 - 30K K_{ab}^4 + 15K (K_{ab}^2)^2 - 20K_{ab}^2 K_{ab}^3 \\ &\quad + 24K_{ab}^5, \end{aligned} \quad (5f)$$

$$\begin{aligned} L_6 &= K^6 - 15K^4 K_{ab}^2 + 40K^3 K_{ab}^3 - 90K^2 K_{ab}^4 + 45K^2 (K_{ab}^2)^2 \\ &\quad - 120K K_{ab}^2 K_{ab}^3 + 144K K_{ab}^5 + 90K_{ab}^2 K_{ab}^4 - 15(K_{ab}^2)^3 + 40(K_{ab}^3)^2 \\ &\quad - 120K_{ab}^6, \\ &= \mathcal{R}^3 - 12\mathcal{R} \mathcal{R}_{ab} \mathcal{R}^{ab} + 16\mathcal{R}_{ab} \mathcal{R}^a_c \mathcal{R}^{bc} + 24\mathcal{R}_{ab} \mathcal{R}_{cd} \mathcal{R}^{acbd} \\ &\quad + 3\mathcal{R} \mathcal{R}_{abcd} \mathcal{R}^{abcd} - 24\mathcal{R}_{ab} \mathcal{R}^a_{cde} \mathcal{R}^{bcde} + 2\mathcal{R}_{abcd} \mathcal{R}^{ab}_{ef} \mathcal{R}^{cdef} \\ &\quad - 8\mathcal{R}_{abcd} \mathcal{R}^a_e{}^c{}_f \mathcal{R}^{bfde}, \end{aligned} \quad (5g)$$

where we have used repeatedly the contracted Gauss-Codazzi integrability condition, $\mathcal{R}_{abcd} = K_{ac} K_{bd} - K_{ad} K_{bc}$ and \mathcal{R}_{abcd} denotes the worldvolume Riemann tensor and $K = g^{ab} K_{ab} = K_{ab}^1$. Note that for even n we can recognize the form of the Gauss-Bonnet (GB) terms (see A.4-A.7) but expressed now in terms of the worldvolume extrinsic curvature. For example, for $n = 0$ we have the DNG Lagrangian, for $n = 2$ we have

the Ricci scalar Lagrangian also named Regge-Teitelboim (RT) model [22, 23, 24], for $n = 4$ we have the standard GB Lagrangian which for $p > 3$ produces non-vanishing equations of motion with ghost-free contribution; in fact, for $p = 3$ the GB combination is a total divergence and it is a topological invariant. On the other side, for odd n the corresponding Lagrangians are seen as the Gibbons-Hawking-York boundary terms which may exist if we have the presence of bulk Lovelock invariants (see Appendix A.1). For example, in the original Lovelock gravity theory L_3 is the appropriate Lagrangian necessary in the variational procedure to compensate the surface term appearing when we consider L_4 as the gravitational theory. In short, for a p -brane there are at most $p + 1$ possible terms leading to second-order equations of motion as we will see below.

Some technical remarks are in order. To avoid ambiguities for possible gauge invariance for the case of odd n Lagrangians, it is necessary to make a choice in the direction for the normal vector to the worldvolume in order to have a theory defined on the right hand side of the worldvolume; thereat, we assume that n^μ is such that it is pointing outward to m . Note that definition (2) coincides with the expression of the determinant of K_{ab} .

2.1. Lovelock brane tensors and equations of motion

By virtue of the properties of the gKd function one can define brane conserved tensors as follows

$$J_{(n)b}^a := \delta_{bb_1 b_2 b_3 \dots b_n}^{aa_1 a_2 a_3 \dots a_n} K^{b_1}_{a_1} K^{b_2}_{a_2} K^{b_3}_{a_3} \dots K^{b_n}_{a_n}. \quad (6)$$

These are symmetric and conserved because $\nabla_a J_{(n)}^{ab} = 0$. This fact is straightforwardly shown by using the properties of the gKd and the Codazzi-Mainardi integrability condition for branes, $\nabla_a K_{bc} = \nabla_b K_{ac}$. Notice that, for a $(p+1)$ -dim worldvolume there are at most an equal number of conserved tensors J^{ab} . Similarly, note the resemblance of this definition with the one for the conserved Lovelock tensors (A.10). As developed above, by expanding out the determinant involved in (6) in terms of minors we obtain a recursion relation

$$\begin{aligned} J_{(n)b}^a &= \left[\delta_b^a \delta_{b_1 b_2 b_3 \dots b_n}^{a_1 a_2 a_3 \dots a_n} - \delta_{b_1}^a \delta_{b b_2 b_3 \dots b_n}^{a_1 a_2 a_3 \dots a_n} + \dots + (-1)^n \delta_{b_n}^a \delta_{b b_1 b_2 \dots b_{n-1}}^{a_1 a_2 a_3 \dots a_n} \right] \times \\ &\quad K^{b_1}_{a_1} K^{b_2}_{a_2} K^{b_3}_{a_3} \dots K^{b_n}_{a_n}, \\ &= \delta_b^a L_n - n K^a_c J_{(n-1)b}^c, \end{aligned} \quad (7)$$

where the properties of the gKd have been used[‡]. The recurrent use of this identity allows us to have an expression for the conserved tensors

$$J_{(n)}^{ab} = \sum_{s=0}^n \frac{(-1)^s n!}{(n-s)!} K_{(s)}^{ab} L_{n-s}, \quad n = 0, 1, 2, 3, 4, \dots, p. \quad (8)$$

[‡] The relation (7) was introduced by mathematicians under the name of Newton transformation [25]. In fact, the framework given in (2) was outlined in [25, 26] but, from our perspective, it lacks of a physical insight.

Here we have adopted the notation: $K_{(0)}^{ab} = g^{ab}$, $K_{(1)}^{ab} = K^{ab}$, $K_{(2)}^{ab} = K^a{}_c K^{bc}$ and so forth. We shall observe how J^{ab} determine the mechanical structure of the Lovelock branes. From Eq. (8) we can compute recursively the first conserved tensors

$$J_{(0)}^{ab} = g^{ab} = -2G_{(0)}^{ab}, \quad (9a)$$

$$J_{(1)}^{ab} = g^{ab} L_1 - K^{ab}, \quad (9b)$$

$$J_{(2)}^{ab} = -2G_{(1)}^{ab}, \quad (9c)$$

$$J_{(3)}^{ab} = g^{ab} L_3 - 3\mathcal{R} K^{ab} + 6K K^a{}_c K^{cb} - 6K^a{}_c K^c{}_d K^{db}, \quad (9d)$$

$$J_{(4)}^{ab} = -2G_{(2)}^{ab}, \quad (9e)$$

$$J_{(5)}^{ab} = g^{ab} L_5 - 5L_4 K^{ab} + 20L_3 K^a{}_c K^{cb} - 60L_2 K^a{}_c K^c{}_d K^{db} \\ + 120L_1 K^a{}_c K^c{}_d K^d{}_e K^{eb} - 120L_0 K^a{}_c K^c{}_d K^d{}_e K^e{}_f K^{fb}, \quad (9f)$$

$$J_{(6)}^{ab} = -2G_{(3)}^{ab}, \quad (9g)$$

where $G_{ab}^{(n)}$ denotes the original Lovelock tensors (see Appendix A for details). In view of these facts $J_{(n)}^{ab}$ are to be referred to as *Lovelock brane tensors*. Note that we already have some familiarity with the conservation property of (6). $J_{(0)}^{ab}$ is conserved because we have a Levi-Civita connection; $J_{(1)}^{ab}$ is conserved due to the contraction of the Codazzi-Mainardi integrability condition for branes whereas $J_{(2)}^{ab}$ is nothing but the worldvolume Einstein tensor which is conserved by the Bianchi identity.

The contraction of Eq. (6) with the extrinsic curvature tensor, by considering Eq. (2), provides an identity among the Lovelock brane tensors and the Lovelock brane Lagrangians

$$J_{(n)}^{ab} K_{ab} = L_{n+1}. \quad (10)$$

As a byproduct, it follows immediately from Eqs. (7) and (10) that $J_{(n)a}^a = (p+1-n)L_n = (N-n-1)L_n$.

The main fact behind the Lagrangians (2) is that their associated equations of motion are of second-order in the derivatives of the embedding functions. To prove this, we shall use the so-called conserved stress tensor associated to the each term in (1) defined as follows [21, 27]

$$f_{(n)}^{a\mu} = (L_n g^{ab} - L_n^{ac} K^b{}_c) e^\mu{}_b + (\nabla_b L_n^{ab}) n^\mu, \quad (11)$$

where $L_n^{ab} := \partial L_n / \partial K_{ab}$. It is conserved in the sense that $\nabla_a f_{(n)}^{a\mu} = 0$. This geometrical object is a powerful tool to study the mechanical content of any brane Σ . In our case, in applying the antisymmetric property of the gKd, from Eqs. (2) and (6) we find that $L_n^{ab} = nJ_{(n-1)}^{ab}$. Notice that in some sense, $J_{(n-1)}^{ab}$ is the derivative of L_n . Thus, we have the following

$$f_{(n)}^{a\mu} = (L_n g^{ab} - nJ_{(n-1)}^{ac} K^b{}_c) e^\mu{}_b = J_{(n)}^{ab} e^\mu{}_b, \quad (12)$$

where we have considered the conservation of $J_{(n)}^{ab}$ and the use of the identity (7). Note further that even though we have a second-order action, $f_{(n)}^{a\mu}$ is only tangential to m . This fact is related to the appearance of second-order equations of motion. On physical grounds, (12) is merely the linear momentum density of Σ whose dynamics is

governed by the action (1) which is invariant under Poincaré transformations in the bulk. Following [21], on classical brane trajectories the vanishing of the normal projection of the conservation law, $n \cdot \nabla_a f_{(n)}^a = 0$, yields the equations of motion for the Lagrangian (2) in the compact form

$$J_{(n)}^{ab} K_{ab} = L_{n+1} = 0, \quad (13)$$

whereas the tangential projection of the vanishing of the divergence of (12) results in a geometrical identity being the conservation of the tensors (6). Obviously, we have only one equation of motion which is of second-order in the field variables. This fact, in particular, means that we have only one physical degree of freedom for this type of branes. Some remarks are in order. First, every solution of the Lovelock gravity equations of motion in vacuum, namely $G_{ab}^{(n)} = 0$, is automatically a particular solution of the corresponding Lovelock brane equations of motion. Second, by virtue of the definition of the extrinsic curvature $K_{ab} = -n \cdot \nabla_a e_b$ we can observe that the equation of motion (13) also can be written as a set of conservation laws

$$\nabla_a (J_{(n)}^{ab} e^\mu_b) = 0, \quad (14)$$

where we identify to the linear momentum density (12) as a conserved current.

3. Born-Infeld type brane action

We consider the dynamical evolution of a brane Σ that follows from the local Born-Infeld type action

$$S[X] = \Lambda \int_m d^{p+1} \xi \sqrt{-\det(g_{ab} + X_{ab})}, \quad (15)$$

where

$$X_{ab} = 2\alpha K_{ab} + \alpha^2 K_a^c K_{cb}. \quad (16)$$

Here, Λ is a constant with dimensions $[L]^{-p-1}$ and α is a concomitant constant with dimensions $[L]$ characterizing the relative weight of the nonlinear terms in the model. At first glance, this second-order action in the embedding variables leads to fourth-order equations of motion but this appearance is however deceptive. Notice that $X_{ab} = X_{ba}$ contrary to the standard BI case where $X_{ab} = F_{ab}$ is the electromagnetic field strength[§]. The main symmetry underlying this action is the invariance under worldvolume reparametrizations. It is expected that for small values of X_{ab} the action (15) will reproduce small correction terms to the DNG model as we will uncover later on.

The action (15) has not been built skilfully in order to work properly since it has a natural geometric interpretation despite its somewhat awkward appearance. Indeed, if we consider from the beginning the embedding

$$x^\mu = Y^\mu(\xi^a) = X^\mu + \alpha n^\mu, \quad (17)$$

[§] In this BI spirit a closed approach for strings with $X_{ab} \propto K_a^{ci} K_{bci}$ was developed in [19] where i keeps track of the number of normal vectors to the worldvolume immersed in a higher dimensional bulk.

which is anchored to the former embedding X^μ , then the tangent vectors becomes $E^\mu_a = e^\mu_a + \alpha K_a^b e^\mu_b$ where we have used the Gauss-Weingarten equation $\nabla_a n^\mu = K_a^b e^\mu_b$. The corresponding induced metric $M_{ab} := \eta_{\mu\nu} E^\mu_a E^\nu_b = g_{ab} + X_{ab}$ leads to the geometric volumen element form (15). The embedding (17) is equivalent to foliate the background spacetime by timelike leaves along the transverse deformations of the worldvolume [7, 8, 28].

The BI type action (15) underlies all Lovelock brane Lagrangians. The key to show this fact is to note that $M_{ab} := g_{ab} + X_{ab} = g_{ac}(g^c_d + \alpha K^c_d)(g^d_b + \alpha K^d_b)$. Hence,

$$\begin{aligned}\sqrt{-\det(M_{ab})} &= \sqrt{-\det[g_{ac}(g^c_d + \alpha K^c_d)(g^d_b + \alpha K^d_b)]}, \\ &= \sqrt{-g} [\det(g^a_b + \alpha K^a_b)].\end{aligned}\quad (18)$$

The action (15) thus can be rewritten in the fashion

$$S[X] = \Lambda \int_m d^{p+1} \xi \sqrt{-g} \mathcal{M}, \quad (19)$$

where $\mathcal{M} := \det(g^a_b + \alpha K^a_b)$. Now, for the rest of the section we will rely heavily on the work developed in Appendix B. By considering Eq. (B.3) we obtain

$$[\det(g^a_b + \alpha K^a_b)] = 1 + \sum_{s=1}^{p+1} f_{(s)}, \quad (20)$$

where

$$f_{(s)} = \frac{1}{s!} \delta_{b_1 b_2 b_3 \dots b_s}^{a_1 a_2 a_3 \dots a_s} f^{b_1}_{a_1} f^{b_2}_{a_2} f^{b_3}_{a_3} \dots f^{b_s}_{a_s}, \quad (21)$$

and $f_{ab} = \alpha K_{ab}$. From Eqs. (B.4-B.7) we have

$$f_{(1)} = \text{Tr}(f) = \alpha K, \quad (22a)$$

$$f_{(2)} = \frac{1}{2} [(\text{Tr}(f))^2 - \text{Tr}(f^2)] = \frac{\alpha^2}{2} \mathcal{R}, \quad (22b)$$

$$\begin{aligned}f_{(3)} &= \frac{1}{6} [(\text{Tr}(f))^3 - 3 \text{Tr}(f^2) \text{Tr}(f) + 2 \text{Tr}(f^3)], \\ &= \frac{\alpha^3}{6} (K^3 - 3K K_{ab}^2 + 2K_{ab}^3),\end{aligned}\quad (22c)$$

$$\begin{aligned}f_{(4)} &= \frac{1}{24} [(\text{Tr}(f))^4 + 8 \text{Tr}(f^3) \text{Tr}(f) - 6 \text{Tr}(f^2)(\text{Tr}(f))^2 + 3(\text{Tr}(f^2))^2 \\ &\quad - 6 \text{Tr}(f^4)], \\ &= \frac{\alpha^4}{24} (\mathcal{R}^2 - 4\mathcal{R}_{ab} \mathcal{R}^{ab} + \mathcal{R}_{abcd} \mathcal{R}^{abcd}),\end{aligned}\quad (22d)$$

$$\begin{aligned}&\vdots \\ f_{(s)} &= \frac{\alpha^s}{s!} L_s,\end{aligned}\quad (22e)$$

where L_s is given by (2). Now, if we set $f_{(0)} = 1$, then Eq. (20) becomes

$$[\det(g^a_b + \alpha K^a_b)] = \sum_{s=0}^{p+1} f_{(s)} = \sum_{s=0}^{p+1} \frac{\alpha^s}{s!} L_s. \quad (23)$$

Inserting this expression into Eq. (18) (or Eq. (15)), after a lengthy but straightforward computation, we finally obtain^{||}

$$\begin{aligned} S[X] &= \Lambda \int_m d^{p+1}\xi \sqrt{-g} \left\{ L_0 + \alpha L_1 + \frac{\alpha^2}{2} L_2 + \frac{\alpha^3}{6} L_3 + \frac{\alpha^4}{24} L_4 + \frac{\alpha^5}{120} L_5 \right. \\ &\quad \left. + \frac{\alpha^6}{720} L_6 + \dots \right\}, \\ &= \Lambda \int_m d^{p+1}\xi \sqrt{-g} \sum_{n=0}^{p+1} \left(\frac{\alpha^n}{n!} \right) L_n, \end{aligned} \quad (24)$$

where the expressions (2) have been invoked. This expansion is quite attractive because it exhibits that each one of the emerging terms leads to second-order equations of motion in the embedding functions and so the action (15). From the above expansion, and Eq. (17) we can infer that when the spacetime coordinates suffer a slight deviation along the direction of the normal n^μ , the ordinary DNG volume element undergoes a deformation becoming in another one that can be expressed as a finite series which involves only to the Lovelock brane Lagrangians. Alike, there is a related construction where the strategy of deform a determinant results useful to obtain ghost-free non-linear massive gravity actions [29]. On the other hand, this choice for the bulk coordinates (17) induces, for small deformations of the brane, certain type of brane bending modes in connection with some scalar fields, namely the Galileons, claimed to explain cosmic acceleration, inflation and dark energy [7, 8, 28, 30, 31, 32, 33, 34, 35, 36], where a Galileon is identified with the position modulus of the brane. In fact, it turns out that in $(3+1)$ dimensions the actions that govern the dynamics of this type of scalar field were determined by the corresponding Lovelock Lagrangians for branes [7, 8, 28].

The combination X_{ab} is not constrained to only have the form (16) so it is reasonable to think in another possible choices for X_{ab} maintaining this BI type structure. Some of them are not so attractive due to their complexity and limitation but others still underlie interesting geometrical information. For example, we can choose $X_{ab} = \alpha K_{ab} + \beta K K_{ab}$. By using (B.9), the cubic expansion in small curvature of (15) casts out only the first four terms of the Lovelock invariants by considering $\beta = \alpha^2/4$ but no more beyond. In this sense the action (15) can be also considered as a minimal Born-Infeld extension of the Lovelock brane theory.

4. Classically equivalent formulations

The non-linear form of the BI type action (15) results inconvenient for many purposes, for example, if one is interested in the path integral treatment for this type of theories or in the discussion of the critical dimension. We turn now to provide some actions that describe intrinsically to Σ , and may play a similar role for the action (15)[¶]. To begin

^{||} The computation is rather involved. Hence, we brought into a play the Cadabra software in order to compute the last terms of this expansion [37, 38].

[¶] See Refs. [39, 40, 41, 42] for developments for DNG extended objects and also for Dp -branes

with, suppose that h_{ab} is an auxiliary intrinsic metric on m , not the induced metric. We can treat h_{ab} as an independent variable, and it is identified as the composite metric M_{ab} only as the solution to the classical equation of motion for h_{ab} as we will see below. Consider the action

$$S[X^\mu, h_{ab}] = \frac{\Lambda}{2} \int_m d^{p+1}\xi \sqrt{-h} [h^{ab} (g_{ab} + X_{ab}) - (p-1)], \quad (25)$$

where h^{ab} denotes the inverse of h_{ab} such that $h^{ac}h_{cb} = \delta^a_b$, $h := \det(h_{ab})$ and X_{ab} is given by (16). Note that only for $p = 1$ (a string) the action (25) is invariant under the Weyl transformation $h_{ab} \rightarrow h'_{ab} = e^{2\omega(\xi^a)}h_{ab}$. The corresponding Euler-Lagrange derivative for h_{ab} implies

$$\frac{1}{2}h_{ab} [h \cdot M - (p-1)] = g_{ab} + X_{ab}, \quad (26)$$

where we have introduced the short-hand notation $h \cdot M := h^{ab}(g_{ab} + X_{ab})$. By taking the determinant of this expression and substituting back into (25), it yields the action (15). Of course, note that if $h_{ab} = M_{ab}$ then (26) becomes an identity. If one attempts to consider the introduction of an independent connection in (25) as an auxiliary variable, say $\tilde{\Gamma}$, it is not necessary as discussed in [19]. Indeed, varying (25) with respect to $\tilde{\Gamma}_{bc}^a$, one is able to show that $\tilde{\Gamma}_{bc}^a = g^{ad}\partial_d X \cdot \partial_b \partial_c X = \Gamma_{bc}^a$, that is, the Christoffel symbols associated to g_{ab} .

Let us now look at another alternative action that is Weyl invariant for any p , given by

$$S[X^\mu, h_{ab}] = \Lambda' \int_m d^{p+1}\xi \sqrt{-\det[h_{ab}(h \cdot M)]}, \quad (27)$$

where the modified tension of the brane is $\Lambda' = \Lambda/(p+1)^{(p+1)/2}$. Observe that under the rescaling $h_{ab} \rightarrow h'_{ab} = e^{2\omega(\xi)}h_{ab}$ this action is Weyl invariant because the combination $h_{ab}h^{cd}$ is so. We now focus on the variation of the action (27) with respect to h_{ab} . This in turn implies

$$h_{ab} = (p+1)(h \cdot M)^{-1} M_{ab}. \quad (28)$$

As before, taking the determinant of this expression and plugging back into Eq. (27) we recover again the action (15). Similarly, when $h_{ab} = M_{ab}$ then (28) becomes an identity. Unfortunately the action (27) is non-polynomial and therefore this possesses severe drawbacks when one try to analyze and quantize it

Finally, we want to mention another equivalent action. Following the line of reasoning similar to the one proposed in [41] we propose the action

$$S[X^\mu, h_{ab}, \varphi] = \frac{\Lambda}{2} \int_m d^{p+1}\xi \sqrt{-h} \left[\varphi^{1-\frac{2}{p+1}} h^{ab} (g_{ab} + X_{ab}) - \varphi (p-1) \right], \quad (29)$$

where φ is a nondynamical auxiliary scalar field. The above action for any p is invariant under the Weyl symmetry

$$h_{ab} \rightarrow h'_{ab} = e^{2\omega} h_{ab}, \quad \varphi \rightarrow \varphi' = e^{-(p+1)\omega} \varphi. \quad (30)$$

The variation of the action (29) with respect to φ implies

$$\varphi^{-\frac{2}{p+1}} (h \cdot M) = p + 1. \quad (31)$$

When we plug back (31) into the action (29) we recover the action (27). Note further that if $h_{ab} = M_{ab}$, which is possible by using the corresponding h_{ab} field equation, we find that $\varphi = 1$ and the action (25) is recovered. We think that this last action deserves a closer examination as it seems reasonable to think that the presence of the scalar field will allow us to make contact with the Galileon theory. It will be considered in a forthcoming development.

5. Conclusions

In this work we have first reviewed the Lovelock brane framework which, for even values mimics the standard Lovelock gravity, while for odd values resembles the counterterms necessary for a well posed variational principle in gravity. We have derived the general equations of motion which were written in terms of conserved brane tensors. We then constructed a Born-Infeld type action which underlies all the Lovelock brane Lagrangians. In particular, the action (15) becomes the action (1) where the constants α_n acquire certain form in terms of the concomitant constant α . We observe that this model is insensitive to any dimension of the worldvolume and it exhibits a natural geometric interpretation. Indeed, we may think of this BI type action as a modification of the DNG action in the sense that it is proportional to the volume as measured now by using a peculiar metric consisting of the induced metric g_{ab} modified by some terms constructed from the extrinsic curvature. In fact, this is nothing but a deformation of the DNG volume element where the resulting volume element casts out the Lovelock brane gravity. Of course we may include matter in our approach, like the electromagnetic interaction via $X_{ab} \rightarrow X_{ab} + F_{ab}$, for example, extending our framework to the DBI approach for Dp -branes. This topic will be reported elsewhere. In addition, some classically equivalent actions to (15) were provided and we analyzed their Weyl invariance through the introduction of auxiliary fields. We hope that this BI type Lovelock action may lead us to an entirely new geometrical standpoint for the understanding of the mechanics of the extended objects.

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Appendix A. Lovelock Gravity Review

We give an overview of the Lovelock gravity theory. This is the most general theory of gravity satisfying the following three conditions: 1. The field equations are written in terms of a symmetric rank-2 tensor. 2. The theory is consistent with the conservation law of the energy-momentum tensor. 3. The theory does not include higher than third order derivatives of the metric. Consider a N -dimensional spacetime manifold \mathcal{M} with metric $g_{\mu\nu}$, ($\mu, \nu = 0, 1, 2, \dots, N-1$). In this N -dimensional spacetime the most general action which maintain the field equations of motion for the metric of second-order, as the Einstein-Hilbert action, is the Lovelock gravity action [4] given by

$$S[g_{\mu\nu}] = \frac{1}{2\kappa_n^2} \int_{\mathcal{M}} d^N x \sqrt{-g} \sum_{n=0}^p a_n \mathcal{L}_n(g_{\mu\nu}, \mathcal{R}_{\mu\nu\alpha\beta}), \quad (\text{A.1})$$

where

$$\mathcal{L}_n(g_{\mu\nu}, R_{\mu\nu\alpha\beta}) = \frac{1}{2^n} \delta_{\mu_1\nu_1\mu_2\nu_2\dots\mu_n\nu_n}^{\alpha_1\beta_1\alpha_2\beta_2\dots\alpha_n\beta_n} \prod_{r=1}^n R^{\mu_r\nu_r}_{\alpha_r\beta_r}, \quad (\text{A.2})$$

and $R_{\mu\nu\alpha\beta}$ is the spacetime Riemann tensor. Here $p := [\frac{N}{2}]$ represents the integer part of $\frac{N}{2}$, a_n and κ_n^2 are constant values. In addition, we have used the generalized Kronecker delta function defined by [43]

$$\delta_{\mu_1\nu_1\mu_2\nu_2\dots\mu_n\nu_n}^{\alpha_1\beta_1\alpha_2\beta_2\dots\alpha_n\beta_n} = \delta_{[\mu_1\nu_1\mu_2\nu_2\dots\mu_n\nu_n]}^{\alpha_1\beta_1\alpha_2\beta_2\dots\alpha_n\beta_n} = \begin{vmatrix} \delta_{\mu_1}^{\alpha_1} & \delta_{\nu_1}^{\alpha_1} & \dots & \delta_{\mu_n}^{\alpha_1} & \delta_{\nu_n}^{\alpha_1} \\ \delta_{\mu_1}^{\beta_1} & \delta_{\nu_1}^{\beta_1} & \dots & \delta_{\mu_n}^{\beta_1} & \delta_{\nu_n}^{\beta_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta_{\mu_1}^{\alpha_n} & \delta_{\nu_1}^{\alpha_n} & \dots & \delta_{\mu_n}^{\alpha_n} & \delta_{\nu_n}^{\alpha_n} \\ \delta_{\mu_1}^{\beta_n} & \delta_{\nu_1}^{\beta_n} & \dots & \delta_{\mu_n}^{\beta_n} & \delta_{\nu_n}^{\beta_n} \end{vmatrix}. \quad (\text{A.3})$$

It must be noted that in N dimensions, all terms for which $n > [N/2]$ are total derivatives and the term $n = N/2$ is the Euler density [4, 5]. Thus, only terms for which $n < N/2$ contribute to the field equations. The first four Lovelock Lagrangians are given by

$$\mathcal{L}_0 = 1, \quad (\text{A.4})$$

$$\mathcal{L}_1 = R, \quad (\text{A.5})$$

$$\mathcal{L}_2 = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \quad (\text{A.6})$$

$$\begin{aligned} \mathcal{L}_3 = & R^3 - 12RR_{\mu\nu}R^{\mu\nu} + 16R_{\mu\nu}R^\mu{}_\alpha R^{\nu\alpha} + 24R_{\mu\nu}R_{\alpha\beta}R^{\mu\alpha\nu\beta} \\ & + 3RR_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 24R_{\mu\nu}R^\mu{}_{\alpha\beta\sigma}R^{\nu\alpha\beta\sigma} + 2R_{\mu\nu\alpha\beta}R^{\mu\nu}{}_{\rho\sigma}R^{\alpha\beta\rho\sigma} \\ & - 8R_{\mu\nu\alpha\beta}R^\mu{}_\rho{}^\alpha{}_\sigma R^{\nu\sigma\beta\rho} \end{aligned} \quad (\text{A.7})$$

The standard variational procedure applied to the action (A.1) casts out the equations of motion

$$\mathcal{G}_{\mu\nu} = \kappa_n^2 T_{\mu\nu}, \quad (\text{A.8})$$

with $T_{\mu\nu}$ being the energy-momentum tensor for matter fields coming from a possible matter action S_{matter} appearing in (A.1) and

$$\mathcal{G}_{\mu\nu} = \sum_{n=0}^p a_n G_{\mu\nu}^{(n)}. \quad (\text{A.9})$$

where the so-called Lovelock tensors $G_{\mu\nu}^{(n)}$ are defined as

$$G_{\mu\nu}^{(n)} = -\frac{1}{2^{n+1}} \delta_{\nu}^{\mu\mu_1\mu_2\dots\mu_n\mu_{n+1}\dots\mu_{2n}} R_{\mu_1\mu_2}^{\nu_1\nu_2} \dots R_{\mu_{2n-1}\mu_{2n}}^{\nu_{2n-1}\nu_{2n}}, \quad (\text{A.10})$$

Explicitly, the first four Lovelock tensors are

$$G_{\mu\nu}^{(0)} = -\frac{1}{2}g_{\mu\nu} = -\frac{1}{2}g_{\mu\nu} \mathcal{L}_0, \quad (\text{A.11})$$

$$G_{\mu\nu}^{(1)} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} \mathcal{L}_1, \quad (\text{A.12})$$

$$G_{\mu\nu}^{(2)} = 2(RR_{\mu\nu} - 2R_{\mu\alpha}R^{\alpha}_{\nu} - 2R_{\mu\alpha\nu\beta}R^{\alpha\beta} + R_{\mu}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}) - \frac{1}{2}g_{\mu\nu}\mathcal{L}_2 \quad (\text{A.13})$$

$$\begin{aligned} G_{\mu\nu}^{(3)} = & 3(R^2R_{\mu\nu} - 4RR_{\mu\alpha}R^{\alpha}_{\nu} - 4R_{\mu\nu}R^{\alpha\beta}R_{\alpha\beta} + 8R_{\mu\alpha}R_{\nu\beta}R^{\alpha\beta} \\ & + 8R_{\mu}^{\alpha}R_{\nu\beta\alpha\gamma}R^{\beta\gamma} + 8R_{\nu}^{\alpha}R_{\mu\beta\alpha\gamma}R^{\beta\gamma} + 2RR_{\mu}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma} \\ & + R_{\mu\nu}R^{\alpha\beta\rho\sigma}R_{\alpha\beta\rho\sigma} - 4R_{\mu\alpha}R_{\nu\beta\rho\sigma}R^{\alpha\beta\rho\sigma} - 4R_{\nu\alpha}R_{\mu\beta\rho\sigma}R^{\alpha\beta\rho\sigma} \\ & - 4R_{\mu\alpha\beta\rho}R_{\nu\sigma}^{\beta\rho}R^{\alpha\sigma} + 8R_{\mu\alpha\nu\beta}R^{\alpha\rho\beta\sigma}R_{\rho\sigma} - 8R_{\mu}^{\alpha\beta\rho}R_{\nu\alpha\beta}^{\sigma}R_{\rho\sigma} \\ & + 2R_{\mu\alpha\beta\rho}R_{\nu}^{\alpha\sigma\lambda}R^{\beta\rho}_{\sigma\lambda} + 8R_{\mu\alpha\beta\rho}R_{\nu\sigma}^{\beta\lambda}R_{\lambda}^{\alpha\rho\sigma} - 4R_{\mu\alpha\nu\beta}R^{\alpha\rho\sigma\lambda}R_{\rho\sigma\lambda}^{\beta} \\ & - 4RR_{\mu\alpha\nu\beta}R^{\rho\sigma} + 8R_{\mu\alpha\nu\beta}R^{\alpha}_{\rho}R^{\beta\rho}) - \frac{1}{2}g_{\mu\nu} \mathcal{L}_3. \end{aligned} \quad (\text{A.14})$$

Appendix A.1. Gibbons-Hawking-York-Myers boundary terms

Now we turn to review briefly the so-called Gibbons-Hawking-York-Myers (GHYM) boundary terms. In order to have a well posed variational principle in Lovelock gravity we must consider appropriate surface terms given by [44, 45, 46, 47]

$$S_b = -2\kappa_n^2 \int_{\partial\mathcal{M}} d^{N-1}x \sqrt{h} \mathcal{L}_b^{(n)}, \quad (\text{A.15})$$

where the Lagrangian surface terms may be written in a tensorial form as follows [47, 46]

$$\begin{aligned} \mathcal{L}_b^{(n)} \sim & \int_0^1 dt \delta_{b_1b_2\dots b_{2n-1}}^{a_1a_2\dots a_{2n-1}} K_{a_1}^{b_1} (\mathcal{R}_{a_2a_3}^{b_2b_3} - 2t^2 K_{a_2}^{b_2} K_{a_3}^{b_3}) \times \\ & \times \dots (\mathcal{R}_{a_{2n-2}a_{2n-1}}^{b_{2n-2}b_{2n-1}} - 2t^2 K_{a_{2n-2}}^{b_{2n-2}} K_{a_{2n-1}}^{b_{2n-1}}). \end{aligned} \quad (\text{A.16})$$

Here $K_{\mu\nu}$ is the extrinsic curvature in the spacelike surface manifold $\partial\mathcal{M}$ and we have suppressed coupling constants. The first three counterterms to the Lovelock Lagrangians take the compact form

$$\begin{aligned} \mathcal{L}_b^{(1)} &= K \\ &= -2G_{\mu\nu}^{(0)} K^{\mu\nu}, \end{aligned} \quad (\text{A.17})$$

$$\begin{aligned} \mathcal{L}_b^{(2)} &= -\frac{1}{3}K^3 + K K_{\mu\nu}^2 - \frac{2}{3}K_{\mu\nu}^3 - 2G_{\mu\nu}^{(1)} K^{\mu\nu} \\ &= J - 2G_{\mu\nu}^{(1)} K^{\mu\nu}, \end{aligned} \quad (\text{A.18})$$

$$\begin{aligned} \mathcal{L}_b^{(3)} &= -\frac{2}{15} [K^5 - 10K^3 K_{\mu\nu}^2 + 20K^2 K_{\mu\nu}^3 + 15K (K_{\mu\nu}^2)^2 - 30K K_{\mu\nu}^4 \\ &\quad - 20K_{\mu\nu}^2 K_{\mu\nu}^3 + 24K_{\mu\nu}^5] - 2G_{\mu\nu}^{(2)} K^{\mu\nu} \\ &= M - 2G_{\mu\nu}^{(2)} K^{\mu\nu}, \end{aligned} \quad (\text{A.19})$$

where $K = g^{\mu\nu} K_{\mu\nu}$ and we have introduced the quantities

$$J = -\frac{1}{3}K^3 + KK_{ab}^2 - \frac{2}{3}K_{ab}^3, \quad (\text{A.20})$$

$$M = -\frac{2}{15} \left[K^5 - 10K^3 K_{ab}^2 + 20K^2 K_{ab}^3 + 15K (K_{ab}^2)^2 - 30K K_{ab}^4 - 20K_{ab}^2 K_{ab}^3 + 24K_{ab}^5 \right]. \quad (\text{A.21})$$

Appendix B. Expansion of the BI type action

The determinant of a $n \times n$ matrix A^a_b in terms of the gKd is

$$A := \det(A^a_b) = \frac{1}{n!} \delta_{b_1 b_2 b_3 \dots b_n}^{a_1 a_2 a_3 \dots a_n} A^{b_1}_{a_1} A^{b_2}_{a_2} A^{b_3}_{a_3} \dots A^{b_n}_{a_n}. \quad (\text{B.1})$$

In addition, the associated inverse matrix can be computed as follows

$$(A^{-1})^a_b = \frac{1}{(n-1)!A} \delta_{bb_2 b_3 \dots b_n}^{aa_2 a_3 \dots a_n} A^{b_2}_{a_2} A^{b_3}_{a_3} \dots A^{b_n}_{a_n}. \quad (\text{B.2})$$

The characteristic determinant of the composite matrix $M^a_b = \delta^a_b + X^a_b$ may be expressed in the form [43]

$$\begin{aligned} M := \det(M^a_b) &= 1 + \sum_{s=1}^n \frac{1}{s!} \delta_{b_1 b_2 b_3 \dots b_s}^{a_1 a_2 a_3 \dots a_s} X^{b_1}_{a_1} X^{b_2}_{a_2} X^{b_3}_{a_3} \dots X^{b_s}_{a_s}, \\ &= 1 + \sum_{s=1}^n X_{(s)}, \end{aligned} \quad (\text{B.3})$$

where $s!X_{(s)} = \delta_{b_1 b_2 b_3 \dots b_s}^{a_1 a_2 a_3 \dots a_s} X^{b_1}_{a_1} X^{b_2}_{a_2} X^{b_3}_{a_3} \dots X^{b_s}_{a_s}$ denotes the determinant of the s -rowed minor. In fact, these minors can be expressed in terms of the traces of the X^a_b matrix. For instance,

$$X_{(1)} = \delta_{b_1}^{a_1} X^{b_1}_{a_1} = X^a_a =: \text{Tr}(X), \quad (\text{B.4})$$

$$\begin{aligned} X_{(2)} &= \frac{1}{2!} \delta_{b_1 b_2}^{a_1 a_2} X^{b_1}_{a_1} X^{b_2}_{a_2} = \frac{1}{2} (\delta_{b_1}^{a_1} \delta_{b_2}^{a_2} - \delta_{b_2}^{a_1} \delta_{b_1}^{a_2}) X^{b_1}_{a_1} X^{b_2}_{a_2}, \\ &= \frac{1}{2} (X^a_a X^b_b - X^a_b X^b_a) = \frac{1}{2} [(\text{Tr}(X))^2 - \text{Tr}(X^2)], \end{aligned} \quad (\text{B.5})$$

$$\begin{aligned} X_{(3)} &= \frac{1}{3!} \delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} X^{b_1}_{a_1} X^{b_2}_{a_2} X^{b_3}_{a_3} \\ &= \frac{1}{6} (\delta_{b_1}^{a_1} \delta_{b_2 b_3}^{a_2 a_3} - \delta_{b_2}^{a_1} \delta_{b_1 b_3}^{a_2 a_3} + \delta_{b_3}^{a_1} \delta_{b_1 b_2}^{a_2 a_3}) X^{b_1}_{a_1} X^{b_2}_{a_2} X^{b_3}_{a_3}, \\ &= \frac{1}{6} [\delta_{b_1}^{a_1} (\delta_{b_2}^{a_2} \delta_{b_3}^{a_3} - \delta_{b_3}^{a_2} \delta_{b_2}^{a_3}) - \delta_{b_2}^{a_1} (\delta_{b_1}^{a_2} \delta_{b_3}^{a_3} - \delta_{b_3}^{a_2} \delta_{b_1}^{a_3}) \\ &\quad + \delta_{b_3}^{a_1} (\delta_{b_1}^{a_2} \delta_{b_2}^{a_3} - \delta_{b_2}^{a_2} \delta_{b_1}^{a_3})] X^{b_1}_{a_1} X^{b_2}_{a_2} X^{b_3}_{a_3}, \\ &= \frac{1}{6} [X^a_a X^b_b X^c_c - 3X^a_b X^b_a X^c_c + 2X^a_b X^b_c X^c_a], \\ &= \frac{1}{6} [(\text{Tr}(X))^3 - 3 \text{Tr}(X^2) \text{Tr}(X) + 2 \text{Tr}(X^3)] \end{aligned} \quad (\text{B.6})$$

$$X_{(4)} = \frac{1}{4!} \delta_{b_1 b_2 b_3 b_4}^{a_1 a_2 a_3 a_4} X^{b_1}_{a_1} X^{b_2}_{a_2} X^{b_3}_{a_3} X^{b_4}_{a_4},$$

$$\begin{aligned}
&= \frac{1}{24} \left[\delta_{b_1}^{a_1} (\delta_{b_2}^{a_2} \delta_{b_3 b_4}^{a_3 a_4} - \delta_{b_3}^{a_2} \delta_{b_2 b_4}^{a_3 a_4} + \delta_{b_4}^{a_2} \delta_{b_2 b_3}^{a_3 a_4}) \right. \\
&\quad - \delta_{b_2}^{a_1} (\delta_{b_1}^{a_2} \delta_{b_3 b_4}^{a_3 a_4} - \delta_{b_3}^{a_2} \delta_{b_1 b_4}^{a_3 a_4} + \delta_{b_4}^{a_2} \delta_{b_1 b_3}^{a_3 a_4}) \\
&\quad + \delta_{b_3}^{a_1} (\delta_{b_1}^{a_2} \delta_{b_2 b_4}^{a_3 a_4} - \delta_{b_2}^{a_2} \delta_{b_1 b_4}^{a_3 a_4} + \delta_{b_4}^{a_2} \delta_{b_1 b_2}^{a_3 a_4}) \\
&\quad \left. - \delta_{b_4}^{a_1} (\delta_{b_1}^{a_2} \delta_{b_2 b_3}^{a_3 a_4} - \delta_{b_2}^{a_2} \delta_{b_1 b_3}^{a_3 a_4} + \delta_{b_3}^{a_2} \delta_{b_1 b_2}^{a_3 a_4}) \right] X^{b_1}_{a_1} X^{b_2}_{a_2} X^{b_3}_{a_3} X^{b_4}_{a_4}, \\
&= \frac{1}{24} \left[X^a_a X^b_b X^c_c X^d_d + 8 X^a_b X^b_c X^c_a X^d_d - 6 X^a_b X^b_a X^c_c X^d_d \right. \\
&\quad \left. + 3 X^a_b X^b_a X^c_d X^d_c - 6 X^a_b X^b_c X^c_d X^d_a \right], \\
&= \frac{1}{24} \left[(\text{Tr}(X))^4 + 8 \text{Tr}(X^3) \text{Tr}(X) - 6 \text{Tr}(X^2) (\text{Tr}(X))^2 \right. \\
&\quad \left. + 3 (\text{Tr}(X^2))^2 - 6 \text{Tr}(X^4) \right]. \tag{B.7}
\end{aligned}$$

On the other hand, for our case of interest it will be useful to obtain the Taylor expansion of the square root of the characteristic determinant (B.3) which can be obtained by using the well known expansion $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots$ for $|x| \leq 1$. Hence,

$$[\det(\delta^a_b + X^a_b)]^{1/2} = 1 + \frac{1}{2} \sum_{s=1}^n X_{(s)} - \frac{1}{8} \left(\sum_{s=1}^n X_{(s)} \right)^2 + \frac{1}{16} \left(\sum_{s=1}^n X_{(s)} \right)^3 - \dots \tag{B.8}$$

Thus, up to $O(K^8)$ we have

$$\begin{aligned}
[\det(\delta^a_b + X^a_b)]^{1/2} &= 1 + \frac{1}{2} \text{Tr}(X) \\
&\quad - \frac{1}{8} [2 \text{Tr}(X^2) - \text{Tr}(X)^2] \\
&\quad + \frac{1}{48} [8 \text{Tr}(X^3) - 6 \text{Tr}(X^2) \text{Tr}(X) + \text{Tr}(X)^3] \\
&\quad - \frac{1}{384} [48 \text{Tr}(X^4) - 32 \text{Tr}(X^3) \text{Tr}(X) \\
&\quad + 12 \text{Tr}(X^2) \text{Tr}(X)^2 - 12 \text{Tr}(X^2)^2 - \text{Tr}(X)^4] \\
&\quad + \frac{1}{3840} [384 \text{Tr}(X^5) - 240 \text{Tr}(X^4) \text{Tr}(X) \\
&\quad + 80 \text{Tr}(X^3) \text{Tr}(X)^2 - 20 \text{Tr}(X^2) \text{Tr}(X)^3 \\
&\quad + 60 \text{Tr}(X) \text{Tr}(X^2)^2 - 160 \text{Tr}(X^2) \text{Tr}(X^3) + \text{Tr}(X)^5] \\
&\quad - \frac{1}{46080} [3840 \text{Tr}(X^6) - 2304 \text{Tr}(X^5) \text{Tr}(X) \\
&\quad - 1440 \text{Tr}(X^4) \text{Tr}(X^2) + 720 \text{Tr}(X^4) \text{Tr}(X)^2 \\
&\quad + 960 \text{Tr}(X^3) \text{Tr}(X^2) \text{Tr}(X) - 160 \text{Tr}(X^3) \text{Tr}(X)^3 \\
&\quad - 640 \text{Tr}(X^3)^2 - 180 \text{Tr}(X^2)^2 \text{Tr}(X)^2 + 120 \text{Tr}(X^2)^3 \\
&\quad + 30 \text{Tr}(X^2) \text{Tr}(X)^4 - \text{Tr}(X)^6] + O(X^7). \tag{B.9}
\end{aligned}$$

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